Binomial Series

Fact (Geometric Series) — $(1-x)^{-1} = 1 + x + x^2 + \cdots$ if $|x| < 1$

Example

Given the fact about geometric series what is the series expansion of:

(a) $(1-x)^{-2}$

(b) $(1-x)^{\frac{1}{2}}$

Example

Compute

- $(1-x)^{-2}$
- $(1-x)^{-3}$
- $(1-x)^{-4}$
- $(1-x)^{-5}$

Example

Compute $\frac{1}{1-x} \cdot \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right)$

Fact (Generalised Binomial Coefficients) —

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} \qquad \text{if } \alpha \in \mathbb{C}$$

Example

Prove that if $n \in \mathbb{Z}$

$$(1+x)^n = \sum_{r=0}^{\infty} \binom{n}{r} x^r \qquad \text{if } |x| < 1$$

Lemma

Prove that $\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$

Fact (Generalised Binomial Theorem) — If $\alpha \in \mathbb{C}$

$$(1+x)^\alpha = \sum_{r=0}^{\infty} \binom{\alpha}{r} x^r = \sum_{r=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-r+1)}{r!} x^r \qquad \text{if } |x| < 1$$

Example

Expand $\sqrt[4]{1+2x}$ as an infinite convergent Binomial series, up to and including the term in x^4. (State the range of values of x for which it is valid).

Example

Expand $\sqrt[3]{8+24x}$ as an infinite convergent Binomial series, up to and including the term in x^3. (State the range of values of x for which it is valid).

Example

By considering $(1-x)^{-1}$ and differentiating, find

$$1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \cdots = \sum_{r=1}^{\infty} r^2 x^{r-1}$$

Example

Hence or otherwise, find $1^2 + 2^2 + 3^2 + \cdots + n^2$